

# Wave Function for the Reissner-Nordström Black-Hole

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## Abstract

We study the quantum behaviour of Reissner-Nordström (RN) black-holes interacting with a complex scalar field. A Maxwell field is also present. Our analysis is based on M. Pollock's [1] method and is characterized by solving a Wheeler-DeWitt equation in the proximity of an apparent horizon of the RN space-time. Subsequently, we obtain a wave-function  $\Psi_{\text{RN}}[M, Q]$  representing the RN black-hole when its charge,  $|Q|$ , is small in comparison with its mass,  $M$ . We then compare quantum-mechanically the cases of (i)  $Q = 0$  and (ii)  $M \geq |Q| \neq 0$ . A special emphasis is given to the evolution of the mass-charge rate affected by Hawking radiation.

## 1 Introduction

Recently, there has been a renewed interest in the canonical quantization of black-hole space-times [1] - [13]. A description of earlier works can be found in ref. [2] and the general purpose of the current line of research is to provide an adequate framework to study the last stages of gravitational collapse [1]–[13]. More precisely, the aim is to obtain a description of quantum black holes that would go beyond a semi-classical approximation, where the background metric is treated classically [14]. A classical Hamiltonian formulation constitutes an essential step in this line and several versions can be found in ref. [1]–[13], [15, 16], with the particularity that several matter Lagrangeans are thoroughly treated in [15, 16]. As far as a quantum analysis is concerned, different perspectives have also been employed:  $r$ -Hamiltonian quantization [3], quantization on the apparent horizon [1, 4, 5, 6], reduced phase space quantization (solving the constraints at classical level and isolating the physical degrees of freedom) [7, 8, 9, 10], and also from Ashtekar variables [11].

In this letter, we will extend M. Pollock's method [1] (which was itself influenced by the prior work of A. Tomimatsu [4] on Schwarzschild space-times) to Reissner-Nordström (RN) black-holes

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and test it in this particular case. As a consequence, we will find a wave function for the RN black-hole, which will have an explicit dependence on its mass  $M$  and the charge  $Q$ .

Our motivations are twofold. On the one hand, a RN black-hole with  $M = |Q|$  has supersymmetric properties [17, 18] while one with  $M > |Q|$  does not. Hence, maybe a wave function for the RN black hole could provide us with crucial insights of how black-hole quantum states in N=2 in supergravity would look like [13]. On the other hand, a wave function  $\Psi_{\text{RN}}[M, Q]$  for the RN black hole could also be used to discern how its mass  $M$  varies with respect to a time coordinate and is influenced by the presence of a charge  $Q$ :  $\dot{M} = f(M, Q)$ . Concerning Schwarzschild black-holes, an expression for the variation of  $M$  with respect to a time coordinate due to the back reaction of the Hawking radiation [19] has been determined semi-classically (see e.g., [19, 14]). This was further validated from quantum gravitational models [1, 4, 5, 6]. A similar treatment regarding charged black holes has been very recently and independently presented in ref. [5, 6]. This particular topic contains significative physical relevance: as a charged black-hole emits radiation then  $M \rightarrow |Q|$  and its temperature  $T \rightarrow 0$ . Hence, its mass evaporation process at this point could stop. The presence of the charge in a relation as  $\dot{M} = f(M, Q)$  may be essential to reproduce this physical effect point to others (see, e.g., ref. [21]).

issues of cosmic censorship [20] and Our approach will have distinct features from those presented in ref. [5, 6]. More precisely, this letter is then organized as follows. In section 2 we briefly summarize the essential elements of the Hamiltonian formulation of charged black-holes in the presence of a matter Lagrangean, which is constituted by complex scalar fields and a Maxwell (electromagnetic) vector field. We then express in section 3 this framework in terms of a RN apparent horizon. Basically, we will employ a RN-Vaidya metric [22, 23, 24] to describe the evaporating black-hole. M. Pollock's method [1] is then appropriately adapted to our case and subsequently used<sup>1</sup> and tested. The relevant dynamical quantities and constraints are then obtained. We keep the complex scalar fields and the vector field but within specific approximation limits<sup>2</sup>. We will solve the constraint equations and obtain wave functions  $\Psi_{\text{RN}}[Q, M]$ , which will constitute solutions of the constraints up to terms of the order of  $Q^2/M^2$ . This is done in section 4, where we will compare the cases of  $Q = 0$  and  $M \geq |Q|$ . We will also draw some comments on how does  $M$  change with respect to a time coordinate. Finally, we present our conclusions in section 5.

## 2 Review of the Hamiltonian formulation of RN black-hole

As mentioned above, we briefly review here the main elements of the Hamiltonian formalism for charged black-holes, following the construction introduced in ref. [15, 16] (see also ref. [1, 4, 5, 6]). The 4-dimensional action has the form

$$S = \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{16\pi} \hat{R} - \frac{1}{4\pi} \hat{g}^{\mu\nu} (\hat{D}_\mu \hat{\psi})^\dagger \hat{D}_\nu \hat{\psi} - \frac{1}{16\pi e^2} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right], \quad (1)$$

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<sup>1</sup>Quite recently, A. Hosoya and in particular I. Oda [5, 6] have also independently analysed Schwarzschild and RN black-holes from a quantum gravitational point of view. They employed the Hamiltonian formulation of ref. [15, 16] but their quantum mechanical analysis followed instead the one introduced by A. Tomimatsu [4].

<sup>2</sup>Namely, when the charge  $|Q|$  is small enough so that  $M + \sqrt{M^2 - Q^2} \simeq 2M - \frac{Q^2}{2M}$ .

where the “overhat” denotes a 4-dimensional variable,  $\hat{\psi}$  is a complex scalar field,  $\hat{A}_\mu$  is the electromagnetic potential and  $\hat{F}_{\mu\nu}$  the corresponding field strength,  $e$  is the electric charge,  $\hat{g}_{\mu\nu}$  is the 4 dimensional space-time metric whose Ricci curvature scalar is  $\hat{R}$ . We use units  $G = \hbar = c = 1$  and the indices  $\mu, \nu, \dots$  take the values 0, 1, 2, and 3 while latin indices  $a, b, \dots$  will take the values 0 and 1. Integration over the angular variables lead to an overall factor of  $4\pi$  from eq. (1).

En route towards our reduced model we further take the following steps. To begin with, our 4-dimesional *spherically symmetric* metric is written as

$$ds^2 = h_{ab}dx^a dx^b + \phi^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2)$$

together with the ADM decomposition

$$h_{ab} = \begin{pmatrix} -\alpha^2 + \frac{\beta^2}{\gamma} & \beta \\ \beta & \gamma \end{pmatrix}, \quad h^{ab} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta}{\alpha^2\gamma} \\ \frac{\beta}{\alpha^2\gamma} & \frac{1}{\alpha} - \frac{\beta^2}{\alpha^2\gamma^2} \end{pmatrix}, \quad (3)$$

where  $\alpha, \beta, \gamma, \phi$  are functions of  $(x^0, x^1) = (\tau, r)$  which will be defined later (see eq. (22)). Hence  $\det(h_{ab}) = -\alpha^2\gamma$ . We also take  $\hat{\psi} = \psi(\tau, r)$  and

$$F_{ab} = \varepsilon_{ab}\sqrt{-h}E; \quad E = (-h)^{-1/2}(\dot{A}_1 - A'_0), \quad (4)$$

$$\hat{D}_a\hat{\psi} = \partial_a\psi + ieA_a\psi; \quad \hat{D}_2\hat{\psi} = \hat{D}_3\hat{\psi} = 0. \quad (5)$$

Furthermore, we have that the trace of the extrinsic curvature is written as

$$K = \frac{\dot{\gamma}}{2\alpha\gamma} - \frac{\beta'}{\alpha\gamma} + \frac{\beta\gamma'}{2\alpha\gamma^2}, \quad (6)$$

and which is present in the gravational part of the action (1) as (cf. ref. [15, 16] for details)

$$\frac{1}{2}\sqrt{-h}\phi^2 R = -\alpha\sqrt{\gamma}K \left( n^a \partial_a \phi^2 \right) + \frac{2\alpha'\phi\phi'}{\sqrt{\gamma}} + \text{total derivatives}. \quad (7)$$

Notice we employ “.”  $\equiv \frac{\partial}{\partial\tau}$  and “'”  $\equiv \frac{\partial}{\partial r}$  and  $n^a$  is the normal unit to the  $x^0 = \text{constant}$  surfaces:

$$n^a = \left( \frac{1}{\alpha}, -\frac{\beta}{\alpha\gamma} \right). \quad (8)$$

The next *significant* step consists in choosing the following coordinate gauge:

$$\alpha = \frac{1}{\sqrt{\gamma}} \Leftrightarrow \sqrt{-h} = 1 \quad (9)$$

(see ref. [1] and in particular [2] for a detailed explanation). Hence, we retrieve the reduced Lagrangean<sup>3</sup>

$$\mathcal{L} = \left\{ 1 + \left[ -\frac{\dot{\phi}^2}{\alpha^2} + 2\beta\dot{\phi}\phi' + \left( \frac{1}{\gamma} - \beta^2 \right) (\phi')^2 \right] \right\}$$

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<sup>3</sup>The Lagrangean (10) contains both  $\alpha$  and  $\gamma$  although the relation (9) states that there is a certain relationship between them. In fact, the reason for the particular form of (10) is *only* to simplify the canonical procedure to derive the Hamiltonian  $H = \alpha\mathcal{H}_0 + \beta\mathcal{H}_1 + A_0\mathcal{H}_2$  (see eq. (11) – (19)).

$$\begin{aligned}
& - [\dot{\gamma} - 2\alpha^{-1}(\alpha\beta)'](\phi\dot{\phi} - \alpha^2\beta\phi\phi' + \alpha\alpha'\phi\phi')\} \\
& + \frac{\phi^2}{\alpha^2}(\dot{\psi} + ieA_0\psi)(\dot{\psi}^\dagger - ieA_0\psi^\dagger) - \left(\frac{1}{\gamma} - \beta^2\right)\phi^2(\psi' + ieA_1\psi)(\psi'^\dagger - ieA_1\psi^\dagger) \\
& - \beta\phi^2\left[(\psi' + ieA_1\psi)(\dot{\psi}^\dagger - ieA_0\psi^\dagger) + (\dot{\psi} + ieA_0\psi)(\psi'^\dagger - ieA_1\psi^\dagger)\right] + \frac{1}{2}\phi^2(\dot{A}_1 - A'_0)^2 \quad (10)
\end{aligned}$$

from which we obtain the canonical momenta

$$\pi_\gamma = -\frac{1}{2}(\phi\dot{\phi} - \alpha^2\beta\phi\phi'), \quad (11)$$

$$\pi_\phi = -\frac{\dot{\phi}}{\alpha^2} + \beta\phi' - \frac{1}{2}(\dot{\gamma} - 2\alpha^{-1}\alpha'\beta - 2\beta')\phi \quad (12)$$

$$\pi_\psi = \frac{\phi^2}{\alpha^2}(\dot{\psi}^\dagger - ieA_0\psi^\dagger) - \beta\phi^2(\psi'^\dagger - ieA_1\psi^\dagger), \quad (13)$$

$$\pi_{\psi^\dagger} = \frac{\phi^2}{\alpha^2}(\dot{\psi} + ieA_0\psi) - \beta\phi^2(\psi' + ieA_1\psi), \quad (14)$$

$$\pi_{A_1} = \phi^2(\dot{A}_1 - A'_0) \quad (15)$$

$$\pi_\alpha = \pi_\beta = \pi_{A_0} = 0. \quad (16)$$

Hence, the constraints can be written as

$$\begin{aligned}
\mathcal{H}_0 &= -2\phi^{-1}\sqrt{\gamma}\pi_\gamma\pi_\phi + 2\phi^{-2}\gamma^{3/2}\pi_\gamma^2 - \frac{1}{2}\left[\sqrt{\gamma} - \frac{\phi'^2}{\sqrt{\gamma}} - 2\alpha'\phi\phi'\right] \\
&+ \frac{\pi_\psi\pi_{\psi^\dagger}}{\sqrt{\gamma}\phi^2} + \frac{\phi^2}{\sqrt{\gamma}}(\psi' + ieA_1\psi)(\psi'^\dagger - ieA_1\psi^\dagger) + \frac{\sqrt{\gamma}}{2\phi^2}\pi_{A_1}^2, \quad (17)
\end{aligned}$$

$$\mathcal{H}_1 = \frac{1}{\gamma}\phi'\pi_\phi - 2\pi'_\gamma\gamma' + \frac{1}{\gamma}(\pi_\psi\psi' + ieA_1\pi_\psi\psi + \pi_{\psi^\dagger}\psi'^\dagger - ieA_1\pi_{\psi^\dagger}\psi^\dagger), \quad (18)$$

$$\mathcal{H}_2 = -ie(\pi_\psi\psi - \psi^\dagger\pi_{\psi^\dagger}) - \pi'_{A_1}. \quad (19)$$

Regarding the quantization of this system, we will analyse it in the vicinity of a suitable apparent horizon [16]. In this case,  $\alpha, \beta, \gamma$  are all finite and non-zero. Self-consistency conditions will be found and/or imposed on the “gravitational” constraints (17), (18) and the gauge constraint (19). This is the subject of the following section.

### 3 Description of a RN Black-Hole in the Apparent Horizon

In this section, we will express the constraints (17), (18) and (19) in terms of dynamical quantities defined at an apparent horizon of the RN black-hole, which is defined by the condition [16]

$$h^{ab}(\partial_a\phi)(\partial_b\phi) = 0 \Leftrightarrow \dot{\phi}(\dot{\phi} - \phi') = 0. \quad (20)$$

Moreover, we will follow M. Pollock’s approach [1], which involves some differences with respect to the method present in ref. [4, 5, 6].

The first element of our (and Pollock's) approach was the introduction of the coordinate gauge choice (9) in the action (10), as described in the previous section. This seems to introduce differences in the Hamiltonian structure. In fact, we have (and in ref. [1] as well) a term  $\alpha' \phi \phi'$  in  $\mathcal{H}_0$  (see eq. [17] above and then the expression for  $W$  in pg. 1180 of ref. [15] and subsequently eq. (8) in that ref. as well] after having employed  $\frac{1}{\sqrt{\gamma}} = \alpha$ . In ref. [5, 6] an integration by parts is employed to get  $\frac{\alpha' \phi \phi'}{\sqrt{\gamma}} \rightarrow -\alpha \left[ \frac{\phi \phi'}{\sqrt{\gamma}} \right]'$  without yet using a gauge condition as (9).

In order to obtain a satisfactory description of an evaporating RN black hole on the apparent horizon, it is more convenient to use a RN - Vaidya metric [22, 23]. This has the general form

$$ds^2 = - \left( 1 - \frac{M(v)}{r} + \frac{Q^2}{r^2} \right) dv^2 + 2dvdr + \phi^2 d\Omega_2^2. \quad (21)$$

Here  $v \equiv \tau + r = t + r^*$  is the advanced null Eddington-Finkelstein coordinate,  $r^*$  is the corresponding "tortoise" coordinate and the relationship between the time  $\tau$  and the time coordinate  $t$  for the standard RN metric<sup>4</sup> is  $\tau = t - r + r^*$ . In addition,  $d\Omega_2^2$  represents the area element of the two-sphere:  $d\theta^2 + \sin^2 \theta d\varphi^2$ . Notice that we are treating  $Q$  as a constant.

At this point we will further include and adapt more elements from the method presented in [1] to our RN black hole case. First, we use the coordinates  $(\tau, r, \theta, \varphi)$ . In terms of the coordinates  $(\tau, r, \theta, \varphi)$  the Vaidya RN metric becomes

$$ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) d\tau^2 + \left( \frac{4M}{r} - \frac{2Q^2}{r^2} \right) d\tau dr + \left( 1 + \frac{2M}{r} - \frac{Q^2}{r^2} \right) dr^2 + \phi^2 d\Omega_2^2, \quad (22)$$

where the quantities  $\alpha, \beta, \gamma$  take the form

$$\alpha = \left( 1 + \frac{2M}{r} - \frac{Q^2}{r^2} \right)^{-1/2}, \quad \beta = \frac{2M}{r} - \frac{Q^2}{r^2}, \quad \gamma = 1 + \frac{2M}{r} - \frac{Q^2}{r^2}. \quad (23)$$

The RN black-hole has two apparent horizons, namely at

$$r_{\pm}(v) = M(v) \pm \sqrt{M^2(v) - Q^2}, \quad (24)$$

and we will henceforth restrict ourselves to the case of  $r_+$ . Secondly, in similarity with the Schwarzschild case we also take  $M \simeq M(\tau)$  in the vicinity of the apparent horizon (see ref. [1, 22, 23, 24] for details).

After some lengthy calculations and using the approximations  $\phi \simeq r, M \simeq M(v) \sim M(\tau)$  we obtain that at the apparent horizon  $r_+$  the following quantities can be *exactly* written as

$$\alpha = \frac{1}{\sqrt{2}}, \quad \beta = 1, \quad \gamma = 2, \quad (25)$$

$$h_{00} = 0, \quad h_{01} = 1, \quad h_{11} = 2, \quad (26)$$

$$h^{00} = -2, \quad h^{01} = 1, \quad h^{11} = 0, \quad (27)$$

$$\beta' = \frac{2}{\rho} \left( \frac{M}{\rho} - 1 \right), \quad (28)$$

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<sup>4</sup>Standard RN metric:  $ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_2^2$ .

$$\alpha' = -\frac{1}{2\sqrt{2}\rho} \left( \frac{M}{\rho} - 1 \right), \quad (29)$$

$$\dot{\gamma} = 2\frac{\dot{M}}{\rho}, \quad (30)$$

$$\gamma' = \frac{2}{\rho} \left( \frac{M}{\rho} - 1 \right), \quad (31)$$

$$\pi'_\gamma = -\frac{1}{4} \left( 1 - \frac{M}{\rho} \right) + \frac{1}{4}, \quad \pi_\gamma = \frac{\rho}{4}, \quad (32)$$

$$\pi_\phi = -\dot{M} - \frac{1}{2} + \frac{3}{2} \frac{M}{\rho}, \quad (33)$$

$$\pi_{A_1} = \rho^2(\dot{\bar{A}}_1 - \bar{A}'_0), \quad (34)$$

$$\pi_\psi = 2\rho^2[\dot{\bar{\psi}}^\dagger - \bar{\psi}'^\dagger - ie(\bar{A}_0 - \bar{A}_1)\bar{\psi}^\dagger], \quad (35)$$

$$\pi_{\psi^\dagger} = 2\rho^2[\dot{\bar{\psi}} - \bar{\psi}' + ie(\bar{A}_0 - \bar{A}_1)\bar{\psi}], \quad (36)$$

which agree with the corresponding expressions in the case of  $Q = 0$  and  $\rho = 2M$  (see ref. [1]). A “overline” means the value of the variable taken at the vicinity of  $r_+$ . Employing (25)-(36), the constraint equations are then written as<sup>5</sup> (dropping the overline “bar” henceforth)

$$\mathcal{H}_0 = \frac{1}{2}\pi_\phi - \frac{1}{4} + \frac{M}{4\rho} - \frac{1}{2\rho^2}\pi_\psi\pi_{\psi^\dagger} - \frac{1}{2\rho^2}\pi_{A_1}^2 - \frac{\rho^2}{2}(\psi' + ieA_1\psi)(\psi'^\dagger - ieA_1\psi^\dagger) \quad (37)$$

$$\mathcal{H}_1 = \frac{1}{2}\pi_\phi + \frac{1}{4} - \frac{3M}{4\rho} + \frac{1}{2}[\pi_\psi(\psi' + ieA_1\psi) + \pi_{\psi^\dagger}(\psi'^\dagger - ieA_1\psi^\dagger)] \quad (38)$$

$$= \frac{1}{2}\pi_\phi + \frac{1}{4} - \frac{3M}{4\rho} + \frac{1}{2}(\pi_\psi\psi' + \pi_{\psi^\dagger}\psi'^\dagger) - \frac{A_1}{2}(\mathcal{H}_2 - \pi'_{A_1}) \quad (39)$$

$$\mathcal{H}_2 = -ie(\psi\pi_\psi - \psi^\dagger\pi_{\psi^\dagger}) - \pi'_{A_1}, \quad (40)$$

where we are employing  $\rho \equiv r = r_+ = M + \sqrt{M^2 - Q^2}$ . Later on, we will use the approximation  $\rho \sim 2M - \frac{Q^2}{2M}$ . In the following and where appropriate, we will be replacing  $\phi \simeq r$  by the preceding expression  $\rho(M)$  at the apparent horizon  $r_+$ . Hence,  $\dot{\phi} = 0$  and  $\phi' \neq 0$  and eq. (20) is satisfied.

It is also worthy to notice the following properties, regarding our model prior to expressing eq. (17) and (18) in quantities evaluated at the apparent horizon  $r_+$ . The “purely” geometric terms in eq. (17), (18) (first, second, third and first, second, respectively) are the same either in the Schwarzschild or RN cases. Moreover, the values of  $\alpha, \beta, \gamma$  at the apparent horizon are also the same either in the Schwarzschild or RN cases. However, the value of their spatial derivatives at  $r_+$  is different (see above and ref. [1]): the presence of the charge  $Q$  induces a different geometry and the rate of change along spatial geodesics is different from the Schwarzschild black-hole. If  $Q = 0$ ,  $\rho = 2M$  then we get a proportionality between those geometrical terms in  $\mathcal{H}_0$  and  $\mathcal{H}_1$  at the apparent horizon, in *total* agreement with ref. [1] (cf. the first, second and third terms in either eq. (37) and (38)). Moreover, we should also stress that the “geometrical” terms in (37)

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<sup>5</sup>Notice that first we ought to find the explicit expressions for the variables  $\alpha, \beta, \gamma, \phi, \psi, \psi^\dagger$ , etc, and their time/spatial derivatives (see e.g., eq. (11)–(16), (23)). *Only afterwards* we calculate their value at the apparent horizon: see eq. (25)–(36). Subsequently, we can then employ these quantities in eq. (17)–(19) and retrieve eq. (37)–(40). as described above.

and (38) are different from the corresponding ones in [5, 6]. We will further discuss this aspect in section 5.

Another important characteristic of the method employed in ref. [1] (see also ref. [1, 23, 24]) is as follows. The RN-Vaidya metric requires (in  $d = 4$  dimensions) a trace-free matter source energy momentum tensor, which in our case involves complex scalar fields  $\psi, \psi^\dagger$  with the  $A_\mu$  field. Hence, we obtain

$$\begin{aligned} T_{matter} = & -\frac{3}{2}(\dot{\psi}^\dagger\dot{\psi} + e^2 A_0^2 \psi^\dagger\psi + ieA_0\dot{\psi}^\dagger\psi - ieA_0\psi^\dagger\dot{\psi}) \\ & + \frac{3}{4}(\dot{\psi}^\dagger\dot{\psi}' - e^2 A_0 A_1 \psi^\dagger\psi + ieA_1\dot{\psi}\dot{\psi}^\dagger - ieA_0\dot{\psi}^\dagger\dot{\psi}'), \\ & + \frac{3}{4}(\dot{\psi}\dot{\psi}'^\dagger - e^2 A_0 A_1 \psi^\dagger\psi + ieA_0\dot{\psi}\dot{\psi}'^\dagger - ieA_1\dot{\psi}^\dagger\dot{\psi}) = 0. \end{aligned} \quad (41)$$

A simple possibility for a boundary condition extracted from (41) is

$$\dot{\psi}^\dagger(\dot{\psi} - \dot{\psi}') + \dot{\psi}(\dot{\psi}^\dagger - \dot{\psi}'^\dagger) = 0, \quad A_0 = 0, \quad (42)$$

which has important similarities to what Pollock [1] introduced for the Schwarzschild case.

Let us also introduce the following re-scaling which will allow us to monitor our case in close comparison with M. Pollock's [1] and also to address the compatibility of these constraints:

$$\psi = \sqrt{2\pi}(\psi_1 + i\chi) \quad (43)$$

$$\pi_\psi \rightarrow \frac{1}{\sqrt{8\pi}}\pi_\psi, \quad (44)$$

and similarly for their conjugates. We then get the approximate expressions at  $r_+$ :

$$\begin{aligned} \mathcal{H}_0 = & \frac{1}{2}\pi_\phi - \frac{1}{8} + \frac{Q^2}{32M^2} - \frac{1}{16\pi}\frac{1}{\rho^2}(\pi_\chi^2 + \pi_{\psi_1}^2) - \rho^2\pi(\psi_1'^2 + \chi'^2) \\ & - \frac{1}{2\rho^2}\pi_{A_1}^2 - \frac{\rho^2}{2}[2\pi e^2 A_1^2(\psi_1^2 + \chi^2)] - 2\pi e\rho^2 A_1(\psi_1\chi' - \chi\psi_1') \end{aligned} \quad (45)$$

$$\mathcal{H}_1 = \frac{1}{2}\pi_\phi - \frac{1}{8} - \frac{3Q^2}{32M^2} + \frac{1}{2}[\pi_{\psi_1}\psi_1' + \pi_\chi\chi'] + \frac{eA_1}{4}(\pi_{\psi_1}\chi - \pi_\chi\psi_1) \quad (46)$$

Concerning the compatibility of the  $\mathcal{H}_0$  and  $\mathcal{H}_1$  constraints, the following points are now in order. As we mentioned earlier, it is worthwhile to notice that as far as the “purely geometrical” terms in (17) and (18) are concerned, an *exact* proportionality  $\mathcal{H}_0 = C\mathcal{H}_1$  can only be achieved if we choose  $\rho = 2M$  ( $Q = 0$ ), where  $C$  is some constant. In our case, however, we can only have an *approximate* proportionality for these terms and to this end we will require a small value for  $|Q|$  in comparison with  $M$ . Then, compatibility for the RN “geometrical” terms above (and let us emphasize, *within the method* of ref. [1] that we are *testing* in the RN case) could be satisfactory up to terms of order  $\frac{Q^2}{M^2}$ , which will be considered as a physical perturbation. We will further take  $\chi$  as a small perturbation and hence will consider any electromagnetic related terms as with a *very* small magnitude.

Moreover, compatibility between the fourth and fifth terms in (45) and the fourth term in (46) requires that (see ref. [1] for a comparison)

$$\pi_{\psi_1} = -4\pi\rho^2\psi_1', \quad \pi_\chi = -4\pi\rho^2\chi'. \quad (47)$$

Quite interestingly, (47) also constitutes the precise compatibility condition that is necessary between the eight term in (45) and the last terms in (46).

Another option at this point is to require  $A_1$  to be small, and then take the terms with  $e\rho A_1\psi_1, e\rho A_1\chi, eA_1\psi_1, eA_1\chi, \frac{\pi_{A_1}}{\rho}$  to be negligible. Subsequently, we could proceed with solving  $\mathcal{H}_0 \simeq \mathcal{H}_1 = 0$ .

With respect to the remaining constraint in eq. (40), notice that the fifth term in eq. (46) can be written basically as  $A_1(\mathcal{H}_2 - \pi'_{A_1})$ , after having employed eq. (39) and (40). Let us now choose to assume that our RN black hole case is such that allows to have<sup>6</sup>  $\mathcal{H}_1 \equiv H_1 \oplus A_1(\mathcal{H}_2 - \pi'_{A_1}) = 0$ , with  $H_1 \gg A_1(\mathcal{H}_2 - \pi'_{A_1}) \simeq 0$ . Note that in our choice of approximation we are taking terms like  $eA_1\chi$  as very small when compared with others as, e.g.,  $\pi_\chi^2$  in (45), and so the fifth term in eq. (46) is being taken as substantially less influential than the others. Remember now that by construction, we have  $\mathcal{H}_2 = 0$ . If we take  $H_1 \gg A_1(\mathcal{H}_2 - \pi'_{A_1}) \simeq 0$  then  $\mathcal{H}_2 = 0$  implies  $\pi'_{A_1} = 0$ . So, within this last restriction we further take  $H_1 = 0$  (as just defined) and also  $\mathcal{H}_2 = \pi'_{A_1} = 0$ .

Let us finally also mention that eq. (47) implies from eq. (41) that at the apparent horizon we may take the conditions  $A_0 = 0$  together with  $\dot{\psi} = \dot{\psi}^\dagger = 0$  (see ref. [1] for the Schwarzschild case). However, we are only imposing these conditions and restrictions *after* varying the action and obtaining the constraint equations. We further take  $A_1 = A_1(r)$ .

## 4 Quantum States from the Wheeler-DeWitt Equation

As described above, the restrictions and boundary conditions introduced in [1] and adapted here to the RN black hole case lead to the approximate Wheeler-DeWitt equation (up to terms in  $Q^2/M^2$  and neglecting terms  $A_1$ -related which are taken of small magnitude) at the apparent horizon  $r_+$ ,

$$\pi_\phi \simeq \frac{\pi_{\psi_1}^2}{4\pi\rho^2(M)} + \frac{\pi_\chi^2}{4\pi\rho^2(M)} + \frac{1}{4}. \quad (48)$$

Quantization proceeds via the operator replacements

$$\pi_\phi \rightarrow -i\frac{\partial}{\partial\phi} \simeq -i\frac{2M^2}{4M^2 + Q^2}\frac{\partial}{\partial M}, \quad (49)$$

$$\pi_{\psi_1} \rightarrow -i\frac{\partial}{\partial\psi_1}, \quad (50)$$

$$\pi_\chi \rightarrow -i\frac{\partial}{\partial\chi}, \quad (51)$$

which yields the Wheeler-DeWitt equation for the wave function  $\Psi$ ,

$$-i\frac{2M^2}{4M^2 + Q^2}\frac{\partial\Psi}{\partial M} = -\frac{1}{16\pi M^2 + \pi Q^4/M^2 - 8\pi Q^2}\left[\frac{\partial^2\Psi}{\partial\psi_1^2} + \frac{\partial^2\Psi}{\partial\chi^2}\right] + \frac{1}{4}\Psi. \quad (52)$$

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<sup>6</sup>This is indeed a *very particular* choice but our physical results (see eq. (52), (55)). In fact, we could alternatively take the other restricting case of  $\psi = \psi_1 + i\chi \rightarrow \psi = \psi_1$  with no gauge constraint (see ref. [5, 6]). We will then obtain solutions like (55) as a generalization of the ones in [1] but *without* the functional  $F$ . The purpose of our choice is just to find a very particular situation of a case where  $\chi$  and  $A_1$  could be present in  $\Psi_{\text{RN}}$ , even if in a very limited case.



Eq. (52) has a Schrödinger-like form and introducing  $\Psi = \hat{\Psi}(M)\tilde{\Psi}(\psi_1, \chi)\zeta(A_1)$  we get the equations

$$\frac{\partial \hat{\Psi}}{\partial M} = \left[ i \frac{8\pi M^4 - 2\pi Q^4 - 4\pi M^2 Q^2 + \pi Q^6/M^2}{32\pi M^4 + 2\pi Q^4 - 16\pi Q^2 M^2} + ik^2 \frac{2M^2 + Q^2}{32\pi M^4 + 2\pi Q^4 - 16\pi Q^2 M^2} \right] \hat{\Psi} \quad (53)$$

$$\frac{\partial^2 \tilde{\Psi}}{\partial \psi_1^2} + \frac{\partial^2 \tilde{\Psi}}{\partial \chi^2} = -k^2 \tilde{\Psi}, \quad (54)$$

whose solutions are

$$\Psi_{\text{RN}}[M, Q; \psi_1, \chi; k] = \Psi_{\text{RN}}^0 e^{i \left[ \frac{1}{4} \left( -\frac{Q}{M} + 2M \right) + k^2 \frac{M}{Q^2 - M^2} \pm 2\sqrt{\pi} k (\psi_1 + \chi) \right]} \zeta(A_1), \quad (55)$$

where  $k^2$  is a separation constant and  $\Psi_{\text{RN}}^0$  an integration constant. There seems to be no adequate procedure to fix the parameter  $k$  without introducing new physics. However, the Schrödinger form of eq. (52) implies the possibility of positive semi-definite probability densities  $\Psi\Psi^*$ . This may suggest that a black hole could evaporate (see below) without violation of unitarity.

In the very particular case of  $\mathcal{H}_2 \simeq \pi'_{A_1} \simeq 0$  then we could take  $\pi_{A_1} \rightarrow -i \frac{\partial}{\partial A_1}$  and get the equation

$$\frac{d}{dr} \left( \frac{\partial \zeta[A_1(r)]}{\partial A_1(r)} \right) = 0, \quad (56)$$

which allows to add (cf. ref. [12])

$$\Psi_{\text{RN}}[M, Q; \psi_1, \chi; k] = e^{i \left[ \frac{1}{4} \left( -\frac{Q}{M} + 2M \right) + k^2 \frac{M}{Q^2 - M^2} \pm 2\sqrt{\pi} k (\psi_1 + \chi) \right]} F \left( \int_{-\infty}^{+\infty} dr A_1 \right), \quad (57)$$

where  $F$  is an arbitrary function. The point to notice is that we now have explicit solutions concerning the dependence in  $M$  and  $Q$  of (55).

It is interesting to notice the following as well. For the Schwarzschild case ( $Q = 0$ ) eq. (57) implies that near to  $M = 0$  the wave function will oscillate with infinite frequency. If  $\dot{M} < 0$ , this would represent the quantum mechanical behaviour of the black hole near the end point of its evaporation. In the RN case, the rapid oscillations will occur again for  $M = 0$  but also when  $M \sim Q$ . I.e., near extremality and when the black hole mass evaporation can eventually stop. Hence, the presence of  $Q$  in  $\Psi_{\text{RN}}$  allowed us to identify some known physical situations of the RN black hole. Moreover, when  $Q \neq 0$  we have more and different values for  $M$  which lead to  $\hat{\Psi} \sim \text{constant}$ .

As far as the mass-charge ratio for the RN black hole is concerned, we will use eq. (33) for  $\pi_\phi$ , i.e.,

$$-i \frac{\partial \Psi_{\text{RN}}}{\partial \phi} = \left[ -\dot{M} - \frac{1}{2} + \frac{3M}{4M^2 - 2Q^2/M} \right] \Psi_{\text{RN}}. \quad (58)$$

From eq. (55) and eq. (58) and get the equation<sup>7</sup>

$$\dot{M} + \frac{1}{4M^2} \frac{a[k^2; Q]}{d(M)} + \frac{1}{4M^4} \frac{b[k^2; Q]}{d(M)} + \frac{c[k^2; Q]}{d(M)M^6} = 0, \quad (59)$$

---

<sup>7</sup>It should be emphasized that at this point we are taking  $M$  as an expectation value (average), i.e., semiclassical value, over the states  $\Psi_{\text{RN}}$ .

where

$$a[k^2; Q] = k^2 - 5Q^2 + Q, \quad (60)$$

$$b[k^2; Q] = k^2 Q^2 - 3Q^4, \quad (61)$$

$$c[k^2; Q] = \frac{k^2 Q^4}{8}, \quad (62)$$

$$d(M) = 1 + \frac{Q^2}{2M^2}. \quad (63)$$

An integration of (59) leads to the approximate result

$$M = \left[ M_0^3 - \frac{3}{2}(k^2 - 5Q^2 + Q) \right]^{1/3} (t - t_0)^{1/3}. \quad (64)$$

We can now identify several physical cases of interest for the RN black hole, according if  $a, b, c, d$  are either positive, zero or negative.

For the case of  $Q = 0$  [1] (Schwarzschild), it is the separation constant  $k^2$  that determines if the black hole is evaporating and decreasing its mass ( $k^2 > 0$ ), or increasing its mass ( $k^2 < 0 \Leftrightarrow k$  imaginary). In the present RN black hole case, the presence of the charge  $Q$  introduces significant changes. In fact, notice that  $d > 0$  and  $c > 0$  (if  $k^2 < 0$ ) but  $a \leq 0$  when  $Q \geq \frac{1+\sqrt{1+20k^2}}{10}$  or  $Q \leq \frac{1-\sqrt{1+20k^2}}{10}$  (if  $k^2 < 0$ ), and  $b \leq 0$  when  $Q \leq -\sqrt{\frac{k^2}{3}}$  or when  $Q \geq \sqrt{\frac{k^2}{3}}$  (if  $k^2 > 0$ ). If  $k^2 < 0$ , then  $b \leq 0$  ( $b = 0 \Leftrightarrow Q = 0$ ) and  $a$  can only be positive if  $1 + 20k^2 > 0$ . When  $a > 0, b > 0$ , then the RN black hole mass decreases (if  $k^2 > 0$ ). In this case, we have an evaporating RN black hole with probability density  $\varrho \equiv \Psi_{\text{RN}} \Psi_{\text{RN}}^* = (\Psi_{\text{RN}}^0)^2$ , independent of  $M, Q, \psi_1$  and  $\chi$ , which expresses the existence of the hole.

A  $\dot{M} > 0$  stage can be obtained from eq. (58), with  $k^2 > 0$  but  $a < 0, b < 0$  and  $c > 0, d > 0$ . If  $Q = 0$ , this possibility is absent. When  $Q \neq 0, k^2 < 0$ , then  $c < 0, b < 0, a < 0$ , if  $1 + 20k^2 > 0$  has real solutions. The latter situation leads to  $\varrho \equiv \Psi_{\text{RN}} \Psi_{\text{RN}}^* = (\Psi_{\text{RN}}^0)^2 e^{-2|k||(\psi+\chi)|}$ , assuming a well behaved solution  $\Psi_{\text{RN}}$ . For the former, we have again that  $\varrho \equiv \Psi_{\text{RN}} \Psi_{\text{RN}}^* = (\Psi_{\text{RN}}^0)^2$ .

## 5 Conclusions and Discussion

The main purpose of this letter was to contribute with an additional and different perspective on the quantization of RN black holes. The physics of black holes is indeed a fascinating subject and we trust our original results may add new information regarding the many approaches to the quantization of charged black holes.

Our canonical quantized RN black hole model has particular characteristics as far as others (see ref. [5, 6]) are concerned. Among these is the fact that we extended the framework introduced in ref. [1] to a RN geometry in the presence of a complex scalar fields and a vector field  $A_\mu$ . The method of A. Tomimatsu [4] was instead used in ref. [5, 6] for the case of the Schwarzschild and RN black holes. Moreover, we employed the Hamiltonian formulation present in [15, 16], which was also used in ref. [5, 6] and with which this work could be compared.

One of the main differences between this letter and ref. [5, 6] corresponds to the expressions for the  $\mathcal{H}_0$  and  $\mathcal{H}_1$  constraints (37) and (38) at the apparent horizon  $r_+$ . Our results were thoroughly checked and confirmed, hence the reason for these differences could be identified with the

restrictions that M. Pollock's approach [1] induces. Namely, the coordinate gauge choices as (9), subsequent restrictions (41), (42) with  $Q = \text{constant}$ , as well as choosing the  $(\tau, r, \theta, \varphi)$  frame. In fact, we mentioned in the beginning of section 3 that one of the main elements in the approach introduced in ref. [1] (and used here) is the introduction of the gauge choice (9) in the action (10). This seemed to introduce differences in the Hamiltonian structure. In fact, we have (and in ref. [1] as well) a term  $\alpha' \phi \phi'$  in  $\mathcal{H}_0$  [see eq. (17) and then the expression for  $W$  in pg. 1180 of ref. [15] and subsequently eq. (8) in that ref. as well] after employing  $\frac{1}{\sqrt{\gamma}} = \alpha$ . In ref. [5, 6] an integration by parts is employed to get  $\frac{\alpha' \phi \phi'}{\sqrt{\gamma}} \rightarrow -\alpha \left[ \frac{\phi \phi'}{\sqrt{\gamma}} \right]'$ .

Nevertheless, let us emphasize again that Pollock's method [1] and Tomimatsu-Hosoya-Oda's [4, 5, 6] give equivalent descriptions in the Schwarzschild case. Only for the RN case there seems to be differences. Thus, our results also constitute a test on the method expressed in [1] and how it can be suited to black hole cases other than the Schwarzschild space-time. Moreover, it can further inform on the scope of validity of different canonical formulations which seem to be equivalent in the Schwarzschild case [1, 4], but apparently lead to some differences in the RN case.

Let us also mention that we used  $\psi = \psi_1 + i\chi$ , where  $\chi$  could be interpreted as less physically relevant than  $\psi_1$  from conditions we ought to impose. In addition, several other restrictions and approximations had to be introduced within the application of ref. [1] method for the RN case as we explained in section 3. Only by doing so we could get an approximated consistency and deal satisfactorily with the quantization at the apparent horizon, but needing the charge  $|Q|$  to be smaller when compared with the mass  $M$ . This poses obvious limits in the validity of our comments for  $M \rightarrow |Q|$  but which are still of some qualitative interest.

As a consequence, we obtain quantum states from a Wheeler-DeWitt (Schrödinger-like) equation. This one is present in eq. (52) and solutions are found in eq. (55). The case (i)  $Q = 0$  corresponds to the Schwarzschild case and the corresponding wave function implies a period of infinitely rapid oscillations near  $M \sim 0$ . In the RN case, this also occurs near  $M \sim 0$  but near  $M \sim |Q|$  as well, i.e., when the RN black hole is approaching extremality. Notice that only in this situation the RN black hole has supersymmetric properties.

Can our model have an interpretation with particles coming out from the hole? We could interpretate our analysis in such a way that it predicts a flow of complex scalar particles coming out from the hole. This conclusion seems natural on grounds that we are assuming a spherically symmetric electromagnetic field. Since the physical photons correspond to transverse wave modes, it seems to me that an assumption of a spherical symmetric electromagnetic field excludes a possibility of photons coming out from the hole. A generalization of this to include some spherical asymmetry to our electromagnetic field might perhaps merit further study in forthcoming publications.

Finally, we found that the RN mass-charge ratio permitted a  $\dot{M} > 0$  and  $\dot{M} < 0$  stages. The latter can be associated with usual black hole mass evaporation while the former may require further analysis. The stage with  $\dot{M} > 0$  could suggest a physical effect in the terms of mass inflation [21] or an (in)direct consequence of it. However, our approach (having followed ref. [1] method's) seems to be of limited validity and these precise claims must be taken with some caution.

Nevertheless, our results do bring additional and complementary information regarding other recent research [5, 6]. In particular, by testing the approach of [1] in RN black holes and identifying its limits of application, the method of Tomimatsu-Hosoya-Oda seems to embrace much more

physical situations of gravitational collapse. There is still the need for further investigations in the topic of black hole quantization, which one hopes may provide some interesting insights on the issue of canonical quantization of black holes but in a  $N = 2$  supergravity theory [13].

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